Acta Crystallographica Section A
Foundations of Crystallography
ISSN 0108-7673

Received 17 July 2000
Accepted 11 September 2000

On the symmetry of 4- to 11-hedra

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A table giving the statistics of the visible symmetry point groups for all non-isomorphic 4- to 8-hedra and simple 9- to 11-hedra is presented. Some disagreements between the previous data are eliminated. The main tendency for n-hedra to be asymmetric with increasing n is briefly discussed.

1. Introduction

A series of crystallographic and mineralogical tasks face the combinatorial problem of polyhedra enumeration. Nowadays, the exact numbers of 4- to 12-hedra and simple (only three edges meet at each vertex) 13-hedra are declared (Engel, 1994). Among quite a number of works on the topic (see Duijvestijn & Federico, 1981, for a bibliography), we distinguish those by Fedorov (1893), Bruckner (1900), Bouwkamp (1946, 1947), Britton & Dunitz (1973) and Federico (1975). As a whole, they contain drawings of all 4- to 8-hedra, 4- to 8-acra (*i.e.* polyhedra with 4 to 8 vertices) and simple 9- and 10-hedra. So, any features of these polyhedra may be easily found.

2. Characterization of polyhedra

Fedorov (1893) and Britton & Dunitz (1973) characterize polyhedra by their visible symmetry (*i.e.* maximal possible symmetry of the polyhedra with the same combinatorial type; the term was introduced by Fedorov) while others prefer the automorphism group orders. The automorphism group plays an essential role in the combinatorial morphology of polyhedra. It can be calculated directly from the edge graph of each polyhedron. The polyhedra of the same combinatorial type and different symmetry may exist if the automorphism group of their edge graph has nontrivial subgroups. The determination of the symmetry point group of a polyhedron requires its definite metrical implementation. Three fundamental theorems form the background of the related theory: two theorems by Steinitz – a graph is 3-polyhedral if and only if it is 3-connected and planar (Steinitz, 1922), and every 2-sphere can be realized as a convex 3-polyhedron (Steinitz & Rademacher, 1934); and the theorem by Barnett & Grunbaum (1970) – the shape of any facet of a 3-polyhedron can be preassigned.

In this paper, we deal with the visible symmetries of the polyhedra that directly relate to the automorphism groups of their edge graphs. An automorphism group order n uniquely corresponds to the symmetry point group when $n = 1$ (asymmetric polyhedra) or $n > 2$ is a prime number. The latter corresponds to the polyhedra of the axial n (L_n) symmetry. In other cases, the visible symmetry point group characterizes a polyhedron more precisely. According to various authors, there are some disagreements between the data on the symmetry and automorphism group orders of polyhedra. That is why we decided (i) to independently generate drawings of 4- to 8-hedra and simple 9- to 11-hedra and define their visible symmetry, (ii) to revise the available data.

To generate polyhedra, the Fedorov (1893) recurrence algorithm explained also by Engel (1994) was used. All the 4- to 8-hedra and simple 9- to 11-hedra are published in Voytekhovsky (1999, 2000). Each polyhedron is drawn in an axonometric projection and described by the facet symbol meaning the numbers of trilinear, quadrilateral, pentagonal *etc.* facets and by its visible symmetry point group. The following errors were found in the previous data (Fig. 1). Fedorov (1893) has given two isomorphic polyhedra IV‴11 and

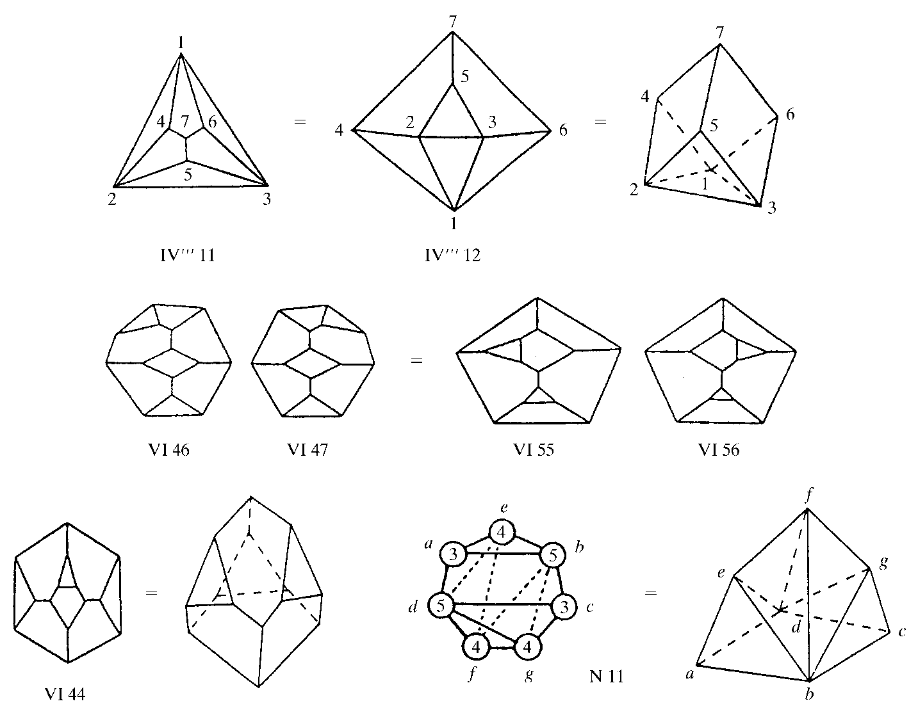

Figure 1
Some errors in determination of the visible symmetry in different projections of the polyhedra. See text for further explanations.

facets:	4	5		6				7					8							9	10	11	
vertices:	4	6	5	8	7	6	5	10	9	8	7	6	12	11	10	9	8	7	6	14	16	18	Total
1									2	3	2		2	21	44	48	22	3		16	137	970	1270
m					1				4	4	2	1	4	12	19	17	11	4		18	51	187	335
2						1		1		2	1		1	1	8	5	5	2		7	18	54	106
3																					1	1	2
$mm2$				1	1			1	2	2		1	3	4	2	1		2	1	5	11	27	64
$2/m$															1						2		3
$3m$								2			2					2				1	6	5	18
$4mm$			1																		1		2
$\bar{4}2m$													1		1		2				1		5
mmm																	1				2		3
$L_5 5P$						1																	1
$\bar{6}m2$		1					1									1				2		4	9
$6mm$											1												1
$\bar{3}m$													1										1
$L_7 7P$																	1						1
$L_{i8} 4L_2 4P$															1						1		2
$L_5 5L_2 6P$								1															1
$\bar{4}3m$	1												1								1		3
$6/mmm$													1										1
$L_7 7L_2 8P$																				1			1
$L_8 8L_2 9PC$																					1		1
$L_9 9L_2 10P$																						1	1
$m\bar{3}m$				1															1				2
Total	1	1	1	2	2	2	1	5	8	11	8	2	14	38	76	74	42	11	2	50	233	1249	1833
	1	2		7				34					257										

Figure 2
The numbers of 4- to 8-hedra and simple 9- to 11-hedra with different visible symmetry point groups. Note: to emphasize non-crystallographic symmetries here and in the text, they are given by the formulae including the axes L_n and inversion axes L_{in} of the nth order, planes P and inversion centre C.

IV‴12 as different ones. He has also duplicated the enantiomorphic pair VI 46, 47 as VI 55, 56. It was Engel (1994) who paid attention to these mistakes for the first time. It may be defined more exactly that Fedorov has erroneously given m symmetry (in recent notation) instead of $3m$ symmetry for the polyhedron IV‴12 as can be seen in an appropriate projection. The numbers make comparison easier. The asymmetric polyhedron VI 46 is equal to 56 while VI 47 is equal to 55. One more error by Fedorov is that the polyhedron VI 44 is not of m but of $3m$ symmetry.

Britton & Dunitz (1973) have erroneously given m symmetry for the polyhedron N 11. From the axonometric projection, it is clear that it is of mm symmetry. The letters make comparison easier. (The authors use the numbers to show how many edges meet at each vertex.) There is one more error in this paper. The polyhedron N 64 has m (C_s) symmetry.

The final results are in Fig. 2. When reduced to the automorphism group orders, these show one more error in Duijvestijn & Federico (1981). The authors give 98 symmetric 10-hedra with 8 of order 6. To get this statistic, drawings in Bruckner (1900) were checked. We found 96 symmetric 10-hedra with 6 of order 6 ($3m$ symmetry). Their numbers in Bruckner (1900) are 15, 34^c, 34^d, 75^b, 75^e, 84. Now, the question on the symmetry of all the 4- to 8-hedra and simple 9- to 11-hedra appears to be exhausted.

The most impressive result is that n-hedra tend to be asymmetric as n increases. In this sense, the variety of 3-polyhedra is mostly asymmetric. As was found by Shafranovsky (1987), Dolivo-Dobrovolsky (1987) and Yushkin (1993), mineral species belonging to the $2/m$ and mmm symmetries prevail in the earth's crust. So, the recent statistics of crystal symmetry are of a different profile. But Khomyakov (1999) predicts a great amount of micro- and nanometric mainly triclinic new mineral species in the future. And the question remains whether the real crystalline polyhedra follow the same tendency as the abstract polyhedra do or not?

To conclude, we would like to look at available data on the 12-hedra. The number of 11-hedra with 12 vertices (64439) does not equal the number of 12-hedra with 11 vertices (64445) as published in Engel (1994). The first number corresponds to that given in Duijvestijn & Federico (1981) and appears to be correct. So, the number of 12-hedra with 11 vertices and, hence, the total number of 12-hedra must be checked. As for simple 12-hedra (7595), to calculate their visible symmetries a computer program was developed and successfully tested for available simple 4- to 11-hedra. Afterwards, the following statistics for simple 12-hedra were found: $1 - 6756$, $\bar{1} - 4$, $m - 597$, $2 - 146$, $3 - 1$, $2/m - 10$, $mm2 - 53$, $222 - 3$, $\bar{4} - 2$, $3m - 5$, $32 - 1$, $\bar{6} - 1$, $mmm - 4$, $\bar{4}2m - 6$, $L_5\,5P - 1$, $\bar{3}m - 2$, $\bar{6}m2 - 1$, $L_{10}\,10L_2\,11P\,C$ (prism) – 1, $15L_2\,10L_3\,6L_5\,15P\,C$ (dodecahedron) – 1. When modified to the automorphism group orders, it corresponds to the data by Duijvestijn & Federico (1981). Nevertheless, this preliminary result should be independently checked.

I acknowledge great benefit from the comments made by two referees. I also thank my student Dmitry G. Stepenshchikov for his assistance in computer searching for simple 12-hedra.

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